# Contents



# Vector Fields

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## 1 Definition of a Vector Field

A vector field over the region $R$ is any function $\mathbf{F}$ that assigns a vector $\mathbf{F}(x, y)$ to every point in the region. Similarly, over a solid region $Q$, $\mathbf{F}$ must assign a vector $\mathbf{F}(x, y, z)$ to every pont in the solid.

A vector field will consist of an infinite number of vectors, although sketching several representative vectors can help show the general behavior of the field.

Given a vector field

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

it is continuous at a point if and only if **all** component functions are continuous at the point.

## 2 Examples of Vector Fields

### 2.1 Velocity Fields

*Velocity fields* describe the motion of particles in a plane or in space, or the flow of fluids through a container or around a moving object.

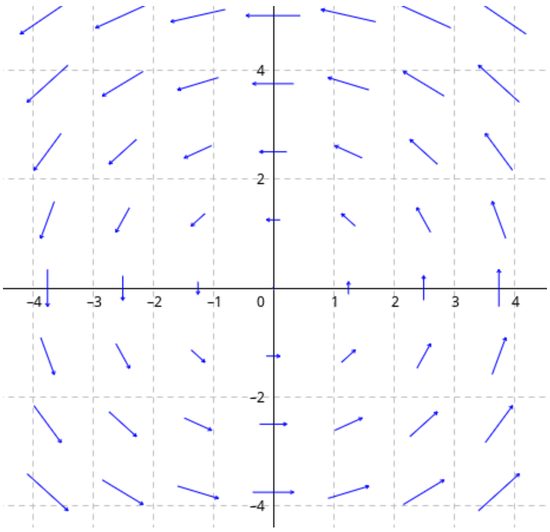


Figure 1: Velocity field about a rotating wheel, $\mathbf{F}(x, y) = \langle -y, x \rangle$

## 2.2 Gravitational Fields

*Gravitational fields* state that the force exterted on a particle of mass $m_1$ at $(x, y, z)$ by another particle with mass $m_2$ at the origin is

$$\mathbf{F}(x, y, z) = \frac{-Gm_1 m_2}{x^2 + y^2 + z^2}\mathbf{u}$$

where $G \approx 6.6743 \times 10^{-11}$. Given a position vector $\mathbf{r} = \langle x, y, z \rangle$, the field can be rewritten as

$$\mathbf{F}(x, y, z) = \frac{-Gm_1 m_2}{\|\mathbf{r}\|^2}\mathbf{u}$$

where $\mathbf{u} = \frac{\mathbf{r}}{\|\mathbf{r}\|}$.

## 2.3 Electric Force Fields

*Electric force fields* state that the force exterted on a particle with charge $q_1$ at $(x, y, z)$ by a particle with charge $q_2$ at the origin is

$$\frac{cq_1 q_2}{\|\mathbf{r}\|^2}\mathbf{u}$$

where $\mathbf{u}$ and $\mathbf{r}$ have the same definitions as they did for gravitational fields. $c$ is a constant that depends on what units are used for $\|\mathbf{r}\|$, $q_1$, and $q_2$.

Note that both electric force fields and gravitational fields have a similar form, called an **inverse square field**.

## Definition of an Inverse Square Field

Let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. The inverse square field of $\mathbf{r}$ is

$$\mathbf{F}(x, y, z) = \frac{k}{\|\mathbf{r}\|^2}\mathbf{u}$$

where $k$ is a real number, and $\mathbf{u} = \frac{\mathbf{r}}{\|\mathbf{r}\|}$.

# 3 Conservative Vector Fields

## 3.1 Gradients, Curl, and Divergence

The gradient operator, $\nabla$, is defined as

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$$

The gradient operator is used in the definition of both curl and divergence, two operations that describe the behavior of a vector field at any given point.

**Divergence** If div $\mathbf{F} = \mathbf{0}$, the function is **divergence free**.

$$\text{div } \mathbf{F}(x, y) = \nabla \cdot \mathbf{F}(x, y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

**Curl** If curl $\mathbf{F} = \mathbf{0}$, the function is **irrotational**.

$$\text{curl } \mathbf{F}(x, y, z) = \nabla \times \mathbf{F}(x, y, z) = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

Given $\mathbf{F}(x, y, z) = \langle M, N, P \rangle$, where all components have continuous second partial derivatives,

$$\text{div}(\text{curl } \mathbf{F}) = \mathbf{0}$$

## 3.2 Finding a Conservative Vector Field

A vector field $\mathbf{F}$ is conservative when there exists another vector field $f$, such that $\nabla f = \mathbf{F}$. The function $f$ is then called a potential function for $\mathbf{F}$.

**Test for Conservative Vector Fields in the Plane**

If $M$ and $N$ have continuous partial derivatives on the open disk $R$, the vector field $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is conservative if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

**Test for Conservative Vector Fields in Space**

If $M$, $N$, and $P$ have continuous partial derivatives in the open sphere $Q$, the vector field $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative if and only if $\mathbf{F}$ is irrotational, that is

$$\text{curl } \mathbf{F} = \mathbf{0}$$

To find the potential function $f(x, y, z)$, you can integrate over all partial derivatives, such as in Section 4.3.

# 4 Examples

## 4.1 Finding the Curl and Divergence of a Vector Field

Find the curl and divergence of $\mathbf{F}(x, y, z) = \langle x^3y^2z, x^2z, x^2y \rangle$ at the point $(2, 1, -1)$.
**Solution**

$$\text{div } \mathbf{F}(x, y, z) = \frac{\partial[x^3y^2z]}{\partial x} + \frac{\partial[x^2z]}{\partial y} + \frac{\partial[x^2y]}{\partial z} = 3x^2y^2z$$
$$3(2^2)(1^2)(-1) = -12$$

The divergence at $\mathbf{F}(2, 1, -1)$ is $-12$.

$$\begin{aligned}\text{curl } \mathbf{F}(x, y, z) &= \left(\frac{\partial[x^2y]}{\partial y} - \frac{\partial[x^2z]}{\partial z}\right)\mathbf{i} - \left(\frac{\partial[x^2y]}{\partial x} - \frac{\partial[x^3y^2z]}{\partial z}\right)\mathbf{j} + \left(\frac{\partial[x^2z]}{\partial x} - \frac{\partial[x^3y^2z]}{\partial y}\right)\mathbf{k} \\ &= (x^2 - x^2)\mathbf{i} - (2xy - x^3y^2)\mathbf{j} + (2xz - 2x^3yz)\mathbf{k} \\ &= (4-4)\mathbf{i} - (2(2)(1) - 2^3 1^2)\mathbf{j} + (2(2)(-1) - 2(2^3)(1)(-1)) \\ &= -(4-8)\mathbf{j} + (-4+16)\mathbf{k} = 4\mathbf{j} + 12\mathbf{k}\end{aligned}$$

The curl at $\mathbf{F}(2, 1, -1)$ is $\langle 0, 4, 12\rangle$.

## 4.2 Testing for Conservative Vector Fields in the Plane

Decide if $\mathbf{F}(x, y) = \langle x^2y, xy\rangle$ is conservative or not.
**Solution**

$$\begin{aligned}\frac{\partial[x^2y]}{\partial y} &= x^2 \\ \frac{\partial[xy]}{\partial x} &= y \\ x^2 &\neq y\end{aligned}$$

Since the two partial derivatives are not equal, the vector field is not conservative.

## 4.3 Finding a Potential Function in Space

Find a potential function for $\mathbf{F}(x, y, z) = \langle 2xy, x^2 + z^2, 2yz\rangle$.

**Solution**
If $f(x, y, z)$ is a function such that $\nabla f(x, y, z) = \mathbf{F}(x, y, z)$, then

$$\begin{aligned}f_x &= 2xy \\ f_y &= x^2 + z^2 \\ f_y &= 2yz\end{aligned}$$

and integrating these in respect to $x$, $y$, and $z$ gives

$$\int 2xy \ \mathrm{d}x = x^2y + g(y, z)$$
$$\int (x^2 + z^2) \ \mathrm{d}y = x^2y + z^2y + h(x, z)$$
$$\int 2yz \ \mathrm{d}z = z^2y + k(x, y)$$

Finally, by comparing these answers, we obtain

$$
\begin{aligned}
g(y, z) &= z^2 y + K \\
h(x, z) &= K \\
k(x, y) &= x^2 y + K
\end{aligned}
$$

leading to the final answer of

$$
f(x, y, z) = x^2 y + z^2 y + K
$$